# 不等均值等方差的标准差UMVUE

Exercise 4 (\ #3 .4). Let $(X_1, \ldots, X_m)$ be a random sample from $N(\mu_x, \sigma_x^2)$ and let $Y_1, \ldots, Y_n$ be a random sample from $N(\mu_y, \sigma_y^2)$. Assume that $X_i$ 's and $Y_j$ 's are independent.
(i) Assume that $\mu_x \in R$, $\mu_y \in R$, $\sigma_x^2 > 0$, and $\sigma_y^2 > 0$. Find the UMVUE's of $\mu_x - \mu_y$ and $(\sigma_x/\sigma_y)^r$, where $r > 0$ and $r < n$.
(ii) Assume that $\mu_x \in R$, $\mu_y \in R$, and $\sigma_x^2 = \sigma_y^2 > 0$. Find the UMVUE's of $\sigma_x^2$ and $(\mu_x - \mu_y)/\sigma_x$.
(iii) Assume that $\mu_x = \mu_y \in R$, $\sigma_x^2 > 0$, $\sigma_y^2 > 0$, and $\sigma_x^2/\sigma_y^2 = \gamma$ is known.

Find the UMVUE of $\mu$.
(iv) Assume that $\mu_x = \mu_y \in R$, $\sigma_x^2 > 0$, and $\sigma_y^2 > 0$. Show that a UMVUE of $\mu$ does not exist.
(v) Assume that $\mu_x \in R$, $\mu_y \in R$, $\sigma_x^2 > 0$, and $\sigma_y^2 > 0$. Find the UMVUE of $P(X_1 \le Y_1)$.
(vi) Repeat (v) under the assumption that $\sigma_x = \sigma_y$.

(ii) 假设 $\mu_x \in R$, $\mu_y \in R$，并且 $\sigma_x^2 = \sigma_y^2 > 0$.求 $\sigma_x^2$ 和 $(\mu_x - \mu_y)/\sigma_x$ 的 UMVUE.

**证明**:

对于参数 $(\mu_x, \mu_y, \sigma_x^2)$，其完全充分统计量为 $(\bar{X}, \bar{Y}, S^2)$，其中

$$S^2 = \frac{1}{m+n-2}\left[\sum_{i=1}^{m}(X_i - \bar{X})^2 + \sum_{j=1}^{n}(Y_j - \bar{Y})^2\right].$$

由于 $(m+n-2)S^2/\sigma_x^2$ 服从卡方分布 $\chi^2_{m+n-2}$，因此 $\sigma_x^2$ 的 UMVUE 为 $S^2$，而 $\sigma_x^{-1}$ 的 UMVUE 为 $\kappa_{m+n-2,-1}S^{-1}$，其中 $\kappa_{m+n-2,-1}$ 是卡方分布的调整因子，用于保证无偏性.
这里

$$\kappa_{m,r} = \frac{m^{r/2}\Gamma\left(\frac{m}{2}\right)}{2^{r/2}\Gamma\left(\frac{m+r}{2}\right)}$$

由于 $\bar{X} - \bar{Y}$ 和 $S^2$ 彼此独立，因此 $\kappa_{m+n-2,-1}(\bar{X} - \bar{Y})/S$ 是 $(\mu_x - \mu_y)/\sigma_x$ 的 UMVUE.